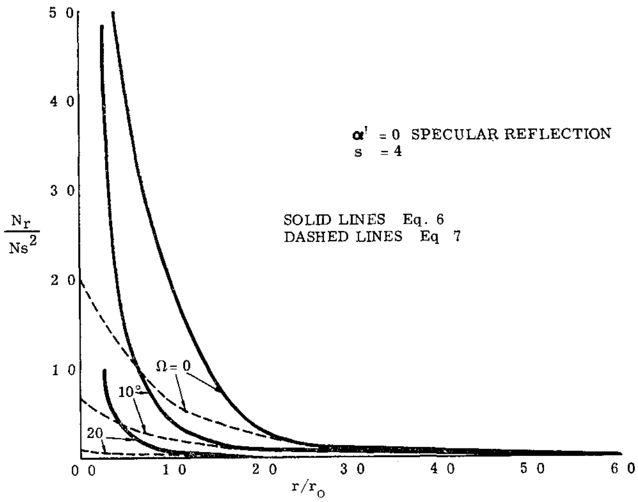

Fig 5 Contours of constant density in the cloud of specularly reflected molecules from a circular plate of radius r_0

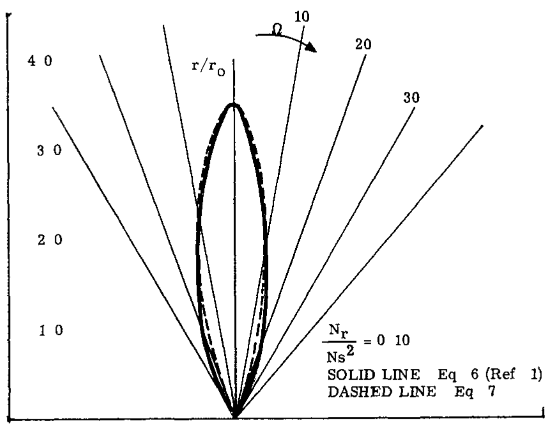

Fig 6 Contours of constant number density in the cloud of specularly reflected molecules from a circular plate of radius r_0

where $a = (r_0/r)^2 + 1$, $b = 2(r_0/r)\sin\Omega$, and K = elliptic integral of the first kind The results of a numerical integration of these integrals are plotted in Fig 2 with

$$\frac{N_r}{Nx(s)}\left(\frac{T_r}{T}\right)^{1/2} \text{ vs } \frac{r}{r_0}$$

for different values of Ω Figure 3 represents the constant density contours in the cloud of diffusely reflected molecules from the plate surface given by both Eqs (1) and (4)

B Specular Reflection ($\alpha' = 0$)

The modified expression for dN_r now becomes[1]

$$dN = \frac{N \sin\theta s^2 \exp(-s^2 \tan^2\Omega')dS}{(1-\alpha')^{1/2}\pi r^2 \cos^3\Omega'} \tag{5}$$

where α' = accommodation coefficient, and Ω' = angle between number density of reflected molecules and the reflected stream direction (Fig 4) Now $\sin\theta = 1$ for the plate set normal to the stream Then

$$dN_r = \frac{Ns^2 \exp(-s^2 \tan^2\Omega')dS}{(1-\alpha')^{1/2}(\pi r^2)\cos^3\Omega'} \tag{6}$$

From Sec A,

$$dS = RdRd\phi \frac{r\cos\Omega}{(R^2 - 2rR\sin\Omega\cos\phi + r^2)^{1/2}}$$

Then

$$N_r = \frac{Ns^2\exp(-s^2\tan^2\Omega)}{\pi(1-\alpha')^{1/2}\cos^2\Omega} \times \int_0^{2\pi}\left\{1 - \frac{1}{[(r_0/r)^2 - 2(r_0/r)\sin\Omega\cos\phi + 1]^{1/2}} + \frac{\sin^2\Omega\cos^2\phi}{1-\sin^2\Omega\cos^2\phi} + \frac{\sin\Omega\cos\phi(r_0/r - \sin\Omega\cos\phi)]}{(1-\sin^2\Omega\cos^2\phi)(\sin\Omega\cos\phi+1)^{1/2}}\right\}d\phi \tag{7}$$

The results are plotted in Figs 5 and 6 where curves obtained from Eq (6), $dS = \pi r_0^2$, are compared to the curves obtained from the more exact equation (7) for $s = 4$ Note that the effect of the finite distribution of elements is much less pronounced for the specular case than for the diffuse reflection

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Trajectories with Constant Normal Force Starting from a Circular Orbit

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The totality of motions for a particle initially in a circular Kepler orbit and acted upon by a constant, in-plane normal force is determined The orbits lie in a ring bounded by two circles, the first with radius equal to the radius of the initial Kepler orbit and the second with radius dependent on the normal force The second circle lies outside the first circle when the normal force is outward and lies inside when the normal force is inward The radius of the second circle cannot exceed twice the radius of the first circle and is reached only when the normal force is 0 230 times the gravity force at the initial radius The point of central attraction is reached only when the normal force is 2 809 times the gravity force at the initial radius The orbit path oscillates periodically between the two circles However, the orbits are not, in general, periodic since they do not close When the magnitude of the normal force is small, the orbits are direct, whereas when the force is large, the orbits are direct near the first circle and retrograde near the second circle

Introduction

A PARTICLE is moving in a circular Kepler orbit when a constant force is applied perpendicular to the instantaneous velocity vector and in the plane of motion What is the resulting motion of the particle?

This problem with the applied force in the normal direction and three other closely related problems with the applied

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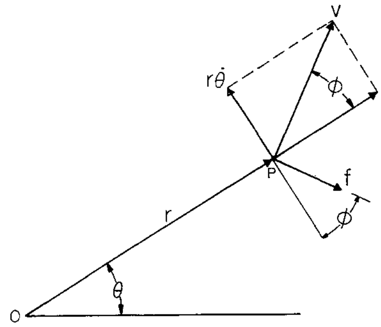

Fig 1 Geometry

force in the radial, circumferential, and tangential directions have received considerable attention during the past ten years Of the four problems, only the radial case has been solved in terms of tabulated integrals (elliptic integrals of the first, second, and third kinds) Next to the radial case, the normal case appears to be the easiest to analyze In spite of this fact, the motion with normal force has not been solved in the sense that Copeland[1] (with corrections by Karrenberg[2] and Au[3]) has solved the radial case

The possibility of reducing the normal case to quadratures was revealed by Rodriquez [4] The complete solution is developed in this paper for the entire range of normal force Since the applied force is perpendicular to the velocity, the energy is conserved Consequently, the semimajor axis of the instantaneous Kepler ellipse (the path which would be traced by the particle if the normal force were removed) is equal to the radius of the initial circular orbit The particle can never move farther than twice the radius of the initial orbit from the point of central attraction

The energy integral may be used to reduce the fourth-order system of equations, which describe the motion, by two orders *A complete reduction to quadratures is possible* when the problem is formulated in plane polar coordinates, but, unlike in the radial case, the quadratures are not tabulated integrals However, even without evaluating the quadratures, the totality of motions can be determined The first step is to determine the angular momentum as an explicit function of the distance from the point of central attraction

Reduction to Quadratures

Let 0 be the point of central attraction and P be the particle The constant normal force f (divided by the mass of P) is positive in the direction shown in Fig 1 The angle between the radial direction and the instantaneous velocity vector v is ϕ The unit of mass is selected so that the universal constant of gravitation is unity The energy and angular momentum are given by

$$E = \tfrac{1}{2}v^2 - (1/r) \qquad h = r^2\dot{\theta}$$

respectively E is a constant but h is not Its time derivative is given by

$$\dot{h} = -fr\cos\phi = -fr\dot{r}/v$$

Using the energy integral to eliminate v, the differential form

$$dh = -fr\{2[E + (1/r)]\}^{-1/2}dr$$

is obtained. Integration in the case when E is negative (corresponding to a Kepler ellipse) provides

$$= \frac{-f}{2(2)^{1/2}E}\left[r^2\left(E+\frac{1}{r}\right)^{1/2}\left(1-\frac{3}{2Er}\right) - \frac{3}{2E}(-E)^{-1/2}\tan^{-1}\left\{\frac{[E+(1/r)]}{-E}\right\}^{1/2}\right] + c$$

where c is the constant of integration

The following analysis is based on the condition that the initial Kepler orbit is circular with radius r_0 E and the initial value for h are given by

$$E = -1/(2r_0) \qquad h_0 = r_0^{1/2}$$

With the nondimensional parameters

$$x = r/r_0 \qquad \alpha = fr_0^2$$

the constant of integration becomes

$$c = r_0^{1/2}\{1 - \alpha[2 + (3\pi/4)]\} \tag{1a}$$

and

$$h = r_0^{1/2}(1 - \alpha U) \tag{1b}$$

where

$$U(x) = 2 + (3\pi/4) - \tfrac{1}{2}(x+3)(2x - x^2)^{1/2} - 3\tan^{-1}[(2/x) - 1]^{1/2}$$

U is imaginary when $x > 2$ The reason is that the semimajor axis of the instantaneous Kepler ellipse would exceed r_0, which is impossible since energy is conserved The critical values for U are

$$U(0) = 2 - (3\pi/4) \approx -0\ 356$$
$$U(1) = 0$$
$$U(2) = 2 + (3\pi/4) \approx 4\ 356$$

U is positive for $x > 1$ and negative for $x < 1$

Having determined h as an explicit function of x, it is possible to obtain the time in quadrature The energy equation may be written as

$$\left(\frac{dr}{dt}\right)^2 = 2\left(E + \frac{1}{r}\right) - \frac{h^2}{r^2} = \left(\frac{2}{r} - \frac{1}{r_0}\right) - \frac{r_0}{r^2}(1 - \alpha U)^2$$

The nondimensional form

$$(dx/d\tau)^2 = (1/x^2)\ [2x - x^2 - (1 - \alpha U)^2] \tag{2}$$

is obtained after multiplying by r_0 and replacing t by $\tau = n_0 t$ where $n_0 = r_0^{-3/2}$ is the mean motion along the instantaneous Kepler ellipse The time quadrature

$$\tau = \int x[2x - x^2 - (1 - \alpha U)^2]^{-1/2}dx + \text{const} \tag{3}$$

is analogous to the Kepler equation for the Kepler problem

The differential equation for the orbit

$$\left(\frac{dx}{d\theta}\right)^2 = \left(\frac{x}{1 - \alpha U}\right)^2 [2x - x^2 - (1 - \alpha U)^2] \tag{4}$$

is obtained from (2) by the operation

$$\frac{dx}{d\theta} = \frac{dx}{d\tau}\frac{d\tau}{d\theta}$$

where

$$\frac{d\theta}{d\tau} = \frac{1}{x^2}(1 - \alpha U) \tag{5}$$

which follows from the definition $h = r^2\dot{\theta}$ The orbit equation is also reducible to quadrature:

$$\theta = \int \frac{1 - \alpha U}{x}\ [2x - x^2 - (1 - \alpha U)^2]^{-1/2}\,dx + \text{const} \tag{6}$$

The quadratures (3) and (6) appear to be intractable Nevertheless, a complete qualitative description of the motion can be obtained without carrying out the integration

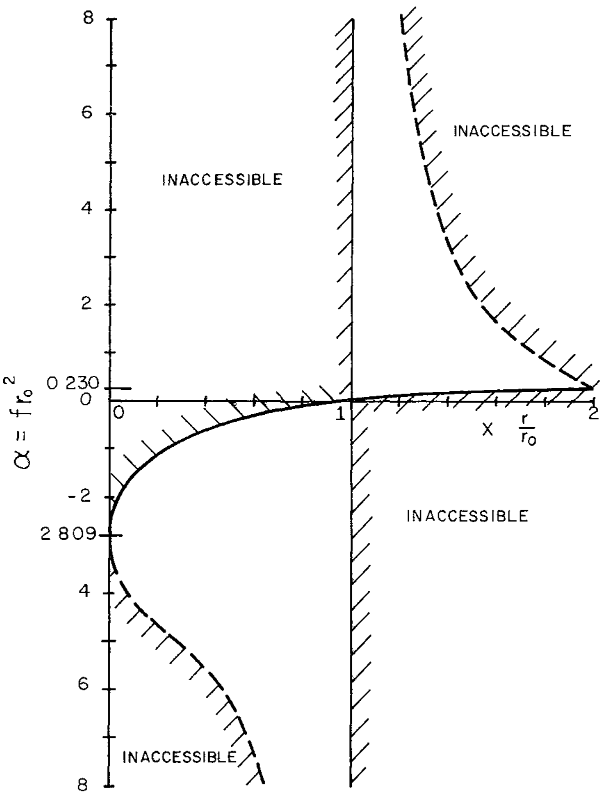

Fig 2 Regions of motion

The Regions of Motion

From Eq (2) the motion is imaginary if

$$f(x) = -(x - 1)^2 + 2\alpha U - \alpha^2 U^2$$

is negative A necessary (though not sufficient) condition for the motion to be real is that αU be positive An examination of the sign of U shows that, if the normal force is directed initially outward (inward), the trajectory will never move interior (exterior) to the initial circular orbit

The question arises whether or not $f(x)$ vanishes at any other value $x = a$ besides 1 Certainly a would depend on α The two roots for α which satisfy the equation $f(x) = 0$ are

$$\alpha(x) = U^{-1}[1 \pm (2x - x^2)^{1/2}] \quad (7)$$

from which

$$\alpha(0) = [2 - (3\pi/4)]^{-1} \approx -2\ 809$$

$$\alpha(1) = \pm \infty$$

$$\alpha(2) = [2 + (3\pi/4)]^{-1} \approx 0\ 230$$

The values for α which satisfy Eq (7) are shown as functions of x in Fig 2 The solid curve in Fig 2 occurs when the minus sign is taken in Eq (7), whereas the two dashed curves occur when the plus sign is used The value for x along these curves is denoted by a For any given value of α, the function $f(x)$ is positive and the motion is ıeal in the region between $x = 1$ and $x = a$ The exterior regions are inaccessible and are so labeled in Fig 2

It is significant that a unique value $\alpha(0)$ is required to reach the origin $x = 0$ and that a second unique value $\alpha(2)$ is required to reach the outer limit $x = 2$ Indeed it is logical to expect the motion to be qualitatively different in the regions $\alpha < \alpha(0)$, $\alpha(0) < \alpha < 0$, $0 < \alpha < \alpha(2)$, and $\alpha > \alpha(2)$

Qualitative Descıiption of the Motion

Considerable information can be obtained by examining Eq (2):

$$(dx/d\tau)^2 = R(x) \qquad R(x) = (1/x^2)\ [2x - x^2 - (1 - \alpha U)^2]$$

$R(x)$ has the following properties: 1) it is continuous; 2) it is zero at $x = 1$ and $x = a$ (the only exception occurs when $a = 0$); 3) it is positive in the region between $x = 1$ and $x = a$; and 4) dR/dx does not vanish at $x = 1$ and $x = a$ Consequently, the trajectory $x = x(\tau)$ has the following characteristics: 1) $x(\tau)$ lies between $x = 1$ and $x = a$ for all values of τ; 2) $dx/d\tau$ only vanishes at $x = 1$ and $x = a$ (however, at $a = 0$ the derivative does not exist); 3) $x(\tau) = x(-\tau)$ when the origin $\tau = 0$ is taken at $x = 1$ or $x = a$ and 4) $x(\tau)$ is periodic with period $2K$, that is, $x(\tau) = x(\tau + 2K)$

It remains to establish the forms for the orbits $x = x(\theta)$ The direction of motion along the boundary $x = 1$ is direct In fact, $d\theta/d\tau = 1$ The direction of motion along the boundary $x = a$ can be established from Eq (5) which provides the following: 1) $d\theta/d\tau > 0$ for $\alpha(0) < \alpha < \alpha(2)$; 2) $d\theta/d\tau < 0$ for $\alpha < \alpha(0)$ and $\alpha > \alpha(2)$; 3) $d\theta/d\tau = 0$ at $\alpha = \alpha(2)$; 4) $d\theta/d\tau = +\infty$ as $\alpha \rightarrow \alpha(0)$ from the positive side; and 5) $d\theta/d\tau = -\infty$ as $\alpha \rightarrow \alpha(0)$ from the negative side An examination of Eq (4) shows that $dx/d\theta$ vanishes at $x = 0$ and $x = a$ except at $a = 2$ where $dx/d\theta$ does not exist

A complete picture of the orbits is given in Fig 3 x is periodic in τ and θ; however, the orbits themselves are not, in general, periodic since they do not close Indeed the only periodic orbits that do exist are isolated The sign of $d\theta/d\tau$ determines whether the motion is direct or retrograde For $\alpha = \alpha(2)$ the orbits have cusps at the outer boundary, and for $\alpha = \alpha(0)$ the orbits pass through 0

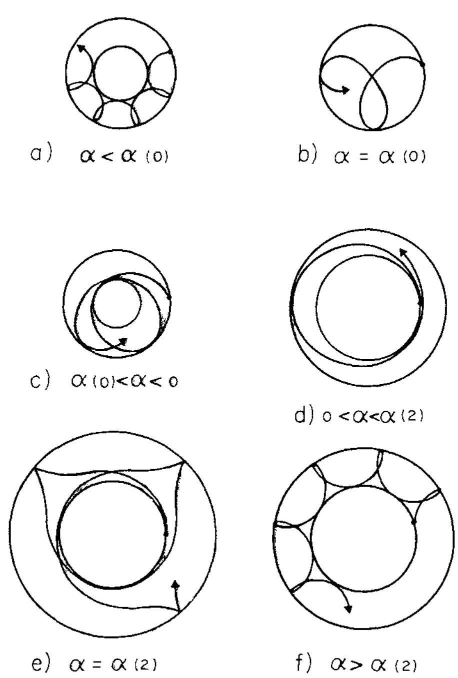

Fig 3 Classification of orbits

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